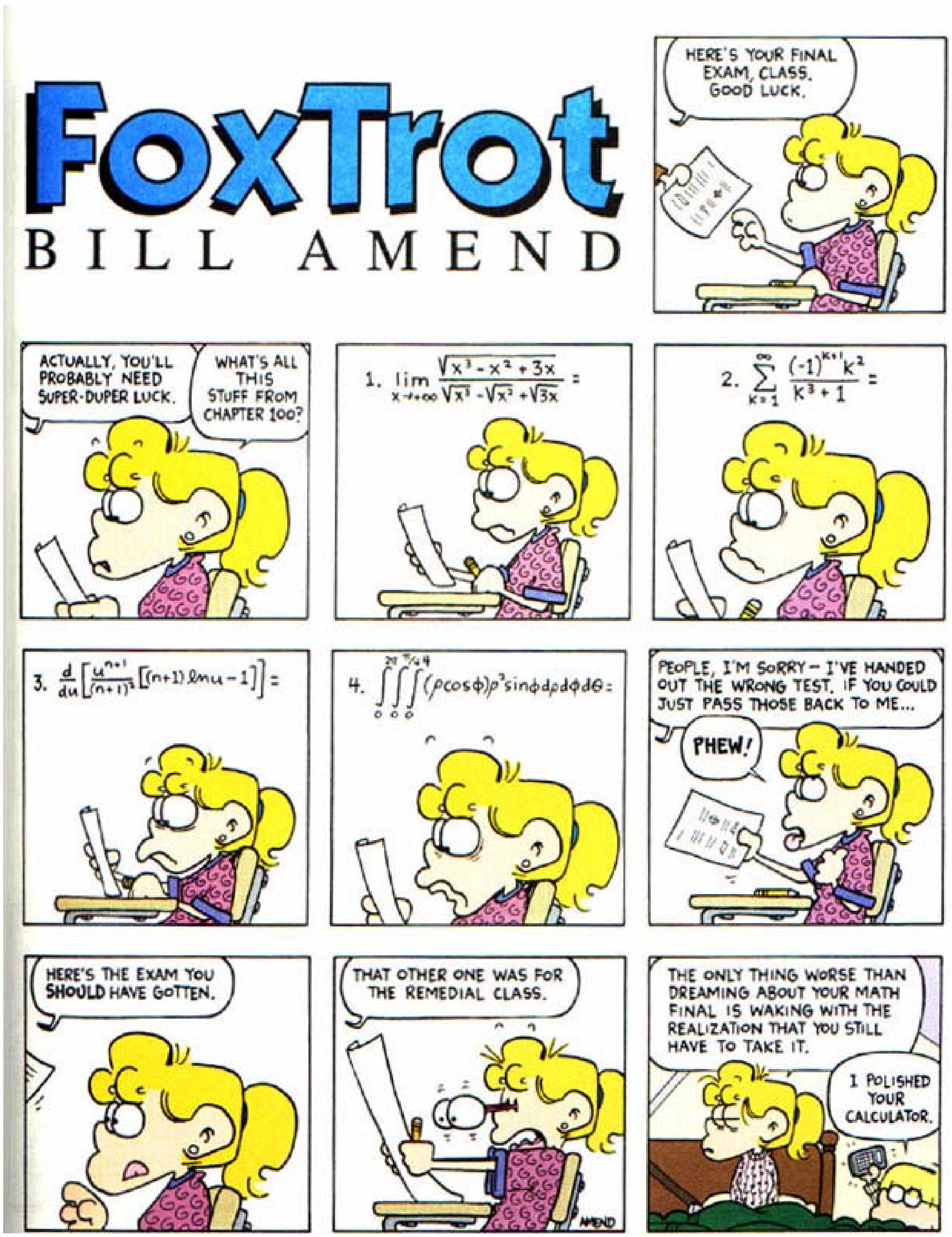
Comic Mathematics

Introduction

Astute artists sometimes provide us with math challenges in their comics. Once the gauntlet is thrown, some will pick up the challenge and solve it. Here is one from Foxtrot by artist Bill Amend dated June 2, 1996.



Foxtrot (four math problems)

Sec 1.1 Problem 1

$$\lim_{x\to\infty}\frac{\sqrt{x^3-x^2+3x}}{\sqrt{x^3}-\sqrt{x^2}+\sqrt{3x}}=??$$

$$\lim_{x\to\infty}\frac{\sqrt{x^3-x^2+3x}}{\sqrt{x^3}-\sqrt{x^2}+\sqrt{3x}}=\lim_{x\to\infty}\frac{\sqrt{x}\left(\sqrt{x^2-x^1+3x^0}\right)}{\sqrt{x}\left(\sqrt{x^2}-\sqrt{x^1}+\sqrt{3x^0}\right)}=\lim_{x\to\infty}\frac{\sqrt{x^2-x^1+3x^0}}{\sqrt{x^2}-\sqrt{x^1}+\sqrt{3x^0}}$$

Squaring numerator and denominator

$$\lim_{x\to\infty}\frac{\sqrt{x^2-x^1+3x^0}\sqrt{x^2-x^1+3x^0}}{\left(\sqrt{x^2}-\sqrt{x^1}+\sqrt{3x^0}\right)\left(\sqrt{x^2}-\sqrt{x^1}+\sqrt{3x^0}\right)}=\lim_{x\to\infty}\frac{x^2-x^1+3x^0}{x^2-2x\sqrt{x}+x\left(1+\sqrt{3}\right)-2\sqrt{3}\sqrt{x}+3}$$

Dividing denominator into the numerator and taking the limit

$$\lim_{x\to\infty}\frac{x^2-x^1+3x^0}{x^2-2x^{3/2}+\left(1+2\sqrt{3}\right)x^1-2\sqrt{3}\,x^{1/2}+3}=\lim_{x\to\infty}\left(1+\frac{2}{x^{1/2}}+\frac{2-2\sqrt{3}}{x^1}-\frac{2-6\sqrt{3}}{x^{3/2}}+\ldots+O\frac{1}{x^5}\right)=1$$

Sec 1.2 Problem 2

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}k^2}{k^3+1}=??$$

Factoring the denominator results in

$$k^3+1=(k+1)\left(k^2-k+1\right)\text{ with roots } k=-1;\quad k=\frac{1\pm i\sqrt{3}}{2}$$

Taking the polynomial in k and solving for the coefficients by partial fractions

$$\frac{k^2}{k^3+1}=\frac{k^2}{\left(k^2-k+1\right)(k+1)}=\frac{Ak}{\left(k^2-k+1\right)}+\frac{B}{\left(k^2-k+1\right)}+\frac{C}{(k+1)}$$

for root $k=-1$

$$\frac{k^2}{\left(k^2-k+1\right)}=C=\frac{(-1)^2}{\left((-1)^2-(-1)+1\right)}=\frac{1}{3}$$

let variable k = 0

$$0 = A(0) + B + C = B + \frac{1}{3}; \quad B = -\frac{1}{3}$$

let variable k = 1

$$\frac{1}{2} = \frac{A}{1} + \frac{B}{1} + \frac{C}{2}; \quad 1 = 2A + 2B + C; \quad A = 1 + \frac{2}{3} - \frac{1}{3} = \frac{2}{3}$$

so

$$\frac{k^2}{(k^2-k+1)(k+1)} = \frac{(2/3)k}{(k^2-k+1)} + \frac{(-1/3)}{(k^2-k+1)} + \frac{1/3}{(k+1)}$$

substituting in the original problem

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1} k^2}{k^3+1} = \sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{2k-1}{3(k^2-k+1)} + \frac{1}{3(k+1)} \right)$$

after a little more work

$$\sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{2k}{3(k^2-k+1)} - \frac{1}{3(k^2-k+1)} + \frac{1}{3(k+1)} \right) = \frac{1}{3}\left(1 - \ln(2) + \pi \operatorname{sech}\left(\frac{\pi\sqrt{3}}{2} \right) \right) \cong 0.239561$$

Wolfram has likewise investigated the solution to this problem.

http://mathworld.wolfram.com/FoxTrotSeries.html

Sec 1.3 Problem 3

$$\frac{d}{du}\left[\frac{u^{n+1}}{(n+1)^2} \left[(n+1)\ln(u) - 1 \right] \right] = ??$$

$$\frac{d}{du}\left[\frac{u^{n+1}}{(n+1)} \left[\ln(u) - \frac{1}{(n+1)} \right] \right] =$$

$$\frac{d}{du}\left[\frac{u^{n+1}}{(n+1)}\ln(u)-\frac{u^{n+1}}{(n+1)^2}\right]=\frac{(n+1)u^n}{(n+1)}\ln(u)+\frac{u^{n+1}}{(n+1)}\frac{1}{u}-\frac{(n+1)u^n}{(n+1)^2}$$

$$\frac{d}{du}[...]=u^n\ln(u)+\frac{u^n}{(n+1)}-\frac{u^n}{(n+1)}=u^n\ln(u)$$

Sec 1.4 Problem 4

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^4(\rho\cos(\phi))\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta=??$$

This triple integral is separable according each variable $d\rho, d\phi, d\theta$.

$$\left(\int_0^{2\pi}1\,d\theta\right)\left(\int_0^{\pi/4}\cos(\phi)\sin(\phi)\,d\phi\right)\left(\int_0^4\rho^3 d\rho\right)=$$

Using substitution for the second integral and normal integration for the first and third.

$$u=\sin(\phi);\quad du=\cos(\phi)\,d\phi;\quad \int_0^{\pi/4}\sin(\phi)\cos(\phi)\,d\phi=\int u\,du=\frac{u^2}{2}=\left.\frac{\sin^2(\phi)}{2}\right|_0^{\pi/4}$$

$$\left(\theta\Big|_0^{2\pi}\right)\left(\left.\frac{\sin^2(\phi)}{2}\right|_0^{\pi/4}\right)\left(\left.\frac{1}{4}\rho^4\right|_0^4\right)=$$

$$(2\pi-0)\left(\frac{1}{2}\sin^2(\pi/4)-\sin^2(0)\right)\left(\frac{1}{4}\left(4^4-0^4\right)\right)=$$

$$(2\pi-0)\left(\frac{1/2}{2}-0\right)\left(\frac{1}{4}(256-0)\right)=$$

$$(2\pi)\left(\frac{1/2}{2}\right)(64)=32\,\pi$$